

Gauge Field Theory of Horizontal Symmetry Generated by a Central Extension of the Pauli Algebra

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The standard model of particle physics is generalized so as to be furnished with a horizontal symmetry generated by an intermediary algebra between simple Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$. Above a certain high energy scale $\tilde{\Lambda}$, the horizontal gauge symmetry is postulated to hold so that the basic fermions, quarks and leptons, form its fundamental triplets, and a triplet and singlet of the horizontal gauge fields distinguish generational degrees of freedom. A horizontal scalar triplet is introduced to make the gauge fields super-massive by breaking the horizontal symmetry at $\tilde{\Lambda}$. From this scalar triplet, there emerge real scalar fields which do not interact with fermions except for neutrino species and may give substantial influence on evolution of the universe. Another horizontal scalar triplet which breaks the electroweak symmetry at a low energy scale $\Lambda \simeq 2 \times 10^2 \text{ GeV}$ reproduces all of the results of the Weinberg-Salam theory, produces hierarchical mass matrices with less numbers of unknown parameters in a unified way and predicts six massive scalar particles, some of which might be observed by the future LHC experiment.

§1. Introduction

The Standard Model of particle physics (SM) still possesses a vast area being full of unsettled problems. It is not clear why quarks and leptons exist in three generations¹⁾ and why they possess characteristic mass spectra with hierarchical structure. The SM has long been suffering the plethora of unknown coupling constants in the Yukawa interaction. To investigate systematically this uncultivated fertile area, which goes customarily by the name of the flavor physics, we develop an exploratory scheme of gauge field theory of a horizontal (H) symmetry in this paper.

As a continuous group for a H symmetry, we adopt here a Lie group $\check{G}(\check{\mathcal{A}})$ generated by a central extension of the Pauli algebra, $\check{\mathcal{A}}$, which was found in analysis of quark mass spectra and applied to characterize the flavor mixing matrices (FMMs)^{1),2)} in the previous papers.^{3),4)} While the Pauli algebra consists of three independent 2×2 matrices, the new algebra $\check{\mathcal{A}}$ which is an intermediary algebra between $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ is composed of four independent 3×3 matrices.

Successes of the SM in low energy physics naturally require that our gauge theory inherits its contents such as the basic fermion fields and the gauge fields for the vertical (V) symmetry. To formulate a simple generalization of the SM with the H symmetry, we add a minimal set of gauge and scalar fields. The theory necessitates a triplet and singlet of gauge fields of the H symmetry, which correspond to four independent generators of the algebra $\check{\mathcal{A}}$. In place of the electroweak (EW) scalar doublet in the SM, two types of scalar triplets of the H symmetry, $\check{\Phi}(x)$ and $\Phi(x)$,

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are postulated to exist in the present formalism.

The H and EW symmetries are broken at two stages with high and low energy scales, $\check{\Lambda}$ and Λ ($\check{\Lambda} \gg \Lambda$). ^{*)} Here we develop a new compound mechanism of symmetry breakdown so as to formulate an effective theory with less numbers of unknown parameters for low energy phenomenology. First, a scale of partial breakdown of symmetry is fixed tentatively by finding a local stationary point of the Higgs potential for the triplet $\check{\Phi}(x)$ ($\Phi(x)$). Then, we make full use of possible freedoms in the decomposition of the triplet $\check{\Phi}(x)$ ($\Phi(x)$) so that the residual symmetry is broken and unphysical modes of component fields are forbidden to appear.

At the high energy scale $\check{\Lambda}$, the triplet $\check{\Phi}(x)$ breaks the H symmetry so that all of the gauge fields are transformed into super-massive vector fields and the neutrinos acquire Majorana masses. From this triplet, there emerge the scalar fields that interact solely with neutrino species. Those fields could have substantial influences on the evolution of the universe.

The horizontal symmetry works to reduce the number of the Yukawa coupling constants down to 4/9 of those of the SM. The triplet $\Phi(x)$ breaking the EW symmetry at the low energy scale, $\Lambda \simeq \Lambda_{\text{EW}} = 10^2 \text{ GeV}$, reproduces all results of the Weinberg-Salam theory and produces mass matrices of Dirac type which possess the same number of unknown parameters with the Yukawa coupling constants and create hierarchical mass spectra. The breakdown of the EW symmetry results in six different types of massive scalar fields. It turns out possible to make all of their masses so large that the flavor changing neutral currents (FCNCs) can be suppressed.

In §2, the algebra $\check{\mathcal{A}}$ and associated group $\check{G}(\check{\mathcal{A}})$ are reconsidered in some detail.³⁾ Contents of bosonic fields being necessary to formulate a gauge field theory of the V and H symmetries are examined in §3. We construct explicitly the Lagrangian density of the theory out of invariants of these symmetries in §4. In §5, remark is given on how to use possible freedoms in the decompositions of the scalar triplets so that unphysical modes are effectively forbidden to appear in the phases of the broken symmetries. Breakdowns of the H symmetry at the high energy scale $\check{\Lambda}$ and the EW symmetry at the low energy scale Λ are investigated, respectively, in §6 and §7. We discuss general remarks of our formalism and point out open problems in §8. Analysis on identical equations among EW and H invariants of the scalar triplets is made in Appendix.

§2. Horizontal symmetry

The central extension of the Pauli algebra, $\check{\mathcal{A}}$, is a closed subalgebra of $\mathfrak{su}(3)$. Four independent generators of the algebra $\check{\mathcal{A}}$ are formed by linear combinations of the Gell-Mann matrices λ_j ($j = 1, 2, \dots, 8$) for $\mathfrak{su}(3)$. Its center is the projective

^{*)} This assumption brings in another sort of the hierarchy problem. We will not deal with it, here, in expectation of future resolution of the celebrated ‘‘Hierarchy’’ puzzle.⁵⁾ One possible resolution is to accept the supersymmetric extension.

element defined by

$$\check{D} = \frac{1}{3}(I + \lambda_1 + \lambda_4 + \lambda_6) = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad (2.1)$$

which possesses the idempotent property $\check{D}^2 = \check{D}$. This is the element known as a *democratic matrix*⁶⁾ which acts to provide the quarks and charged leptons with hierarchical mass spectra. Three other linearly independent generators of the algebra are given as follows:

$$\left\{ \begin{array}{lll} \check{\tau}_1 & = & \frac{1}{\sqrt{3}}(\lambda_3 - \lambda_4 + \lambda_6) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \\ \check{\tau}_2 & = & \frac{1}{\sqrt{3}}(\lambda_2 - \lambda_5 + \lambda_7) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}, \\ \check{\tau}_3 & = & \frac{1}{3}(-2\lambda_1 + \lambda_4 + \lambda_6 + \sqrt{3}\lambda_8) = \frac{1}{3} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \end{array} \right. \quad (2.2)$$

which obey the Pauli-type product rules

$$\check{\tau}_j \check{\tau}_k = \delta_{jk} (I - \check{D}) + i \epsilon_{jkl} \check{\tau}_l \quad (2.3)$$

and the normalization condition $\text{Tr}(\check{\tau}_j \check{\tau}_k) = 2\delta_{jk}$. These elements and the democratic element \check{D} are orthogonal in the sense that $\check{D} \check{\tau}_j = \check{\tau}_j \check{D} = 0$. Here, we identify the Lie algebra which generates the basic H symmetry for the flavor physics with the central extension of the Pauli algebra constructed as

$$\check{\mathcal{A}} = \{\check{D}, \check{\tau}_1, \check{\tau}_2, \check{\tau}_3\}. \quad (2.4)$$

The Lie group $\check{G}(\check{\mathcal{A}})^*$ is defined by the set of exponential mappings of all possible linear combinations of the elements of the algebra $\check{\mathcal{A}}$ as follows:

$$\check{G}(\check{\mathcal{A}}) = \left\{ \Omega(\vartheta) = \exp \left(i\vartheta_0 \check{D} + i \sum_{j=1}^3 \vartheta_j \check{\tau}_j \right) : \vartheta_0, \vartheta_j \in \mathbb{R} \right\}. \quad (2.5)$$

This Lie group $\check{G}(\check{\mathcal{A}})$ which is postulated to generate the H symmetry group possesses two subgroups

$$SU_H(2) = \left\{ \Omega_2(\vartheta) = \exp \left(i \sum_{j=1}^3 \vartheta_j \check{\tau}_j \right) : \vartheta_j \in \mathbb{R} \right\} \quad (2.6)$$

*) The group $\check{G}(\check{\mathcal{A}})$ which is generated by linearly independent elements extracted from 3×3 matrix representations of the discrete S_3 symmetry³⁾ can be interpreted as a continuous quantum extension of the discrete classical group S_3 .

and

$$U_H(1) = \left\{ \Omega_1(\vartheta_0) = \exp(i\vartheta_0 \check{D}) : \vartheta_0 \in \mathbb{R} \right\}. \quad (2.7)$$

For analyses below, it is convenient to use the following expression of the group elements in terms of the elements of the algebra $\check{\mathcal{A}}$ as

$$\Omega_2(\vartheta) = \check{D} + \cos \Theta (I - \check{D}) + \frac{\sin \Theta}{\Theta} \left(i \sum_{j=1}^3 \vartheta_j \check{\tau}^j \right), \quad \Omega_1(\vartheta_0) = e^{i\vartheta_0} \check{D} + (I - \check{D}) \quad (2.8)$$

where $\Theta^2 = \sum_{j=1}^3 \vartheta_j^2$.

From explicit representations of the matrices $\check{\tau}_j$ in (2.2), we obtain the relation of complex conjugation, $\check{\tau}_2(i\check{\tau}_j)^* = (i\check{\tau}_j)\check{\tau}_2$, which leads readily to $\check{\tau}_2(\Omega_2(\vartheta))^* = \Omega_2(\vartheta)\check{\tau}_2$. As for the transposition, care must be taken to distinguish matrices and multiplets of the H symmetry from those of the EW and Lorentz symmetries^{*)}. Here, in addition to the ordinary symbol t which is used to transpose the quantities related with the EW and the Lorentz degrees of freedom, a new symbol \check{t} is introduced for the transposition operation for the H symmetric degrees of freedom. From (2.2), one can prove the relation $\check{\tau}_2 \check{t}(i\check{\tau}_j) = (-i\check{\tau}_j)\check{\tau}_2$ which results in the identity $\check{\tau}_2 \check{t} \Omega_2(\vartheta) = \Omega_2(-\vartheta)\check{\tau}_2 = (\Omega_2(\vartheta))^{-1} \check{\tau}_2$. Note that the Pauli matrices τ_j and the matrices $\check{\tau}_j$ in (2.2) have the same behaviors under the operations of complex conjugate and transposition.

The simultaneous eigenstates of the generators $\check{\tau}_3$ and \check{D} are found in the forms

$$|1\rangle = \frac{1}{\sqrt{2}} \check{t}(1, -1, 0), \quad |2\rangle = \frac{1}{\sqrt{6}} \check{t}(1, 1, -2), \quad |3\rangle = \frac{1}{\sqrt{3}} \check{t}(1, 1, 1). \quad (2.9)$$

Note that the doublet $\{|1\rangle, |2\rangle\}$ and singlet $\{|3\rangle\}$ are irreducible representations of the discrete permutation symmetry S_3 . These eigenstates form a convenient basis to represent fundamental multiplets of the H symmetry. Evidently, $\{|1\rangle, |2\rangle\}$ and $\{|3\rangle\}$ are, respectively, the kernels of the generator \check{D} and the set of the generators $\check{\tau}_1, \check{\tau}_2$ and $\check{\tau}_3$. We utilize, in later argument, the fact that the state $|3\rangle$ is an eigenstate for all of the elements of the $\check{G}(\check{\mathcal{A}})$ group as

$$\Omega(\vartheta)|3\rangle = \Omega_2(\vartheta)\Omega_1(\vartheta_0)|3\rangle = e^{i\vartheta_0}|3\rangle \quad (2.10)$$

where ϑ_j ($j = 0, \dots, 3$) are arbitrary real numbers.

To illustrate a unique feature of the H symmetry, let us consider here an arbitrary H triplet $T(=|T\rangle)$ and introduce an operation $\{\!\!\{\dots\}\!\!\}$ on T by

$$\{\!\!\{T\}\!\!\} = \sum_{i=1}^3 T_i = \sqrt{3} \langle 3|T\rangle. \quad (2.11)$$

Then, owing to the property of the vector $|3\rangle$ in (2.10), the group action $T \rightarrow \Omega(\vartheta)T$ induces the transformation $\{\!\!\{T\}\!\!\} \rightarrow \{\!\!\{\Omega(\vartheta)T\}\!\!\} = e^{i\vartheta_0} \{\!\!\{T\}\!\!\}$ on $\{\!\!\{T\}\!\!\}$. Therefore, the quantity $\{\!\!\{T\}\!\!\}$, which we call hereafter the H-sum of T , behaves as an eigenvector for an arbitrary elements of the $\check{G}(\check{\mathcal{A}})$ group. In particular, the H-sum is invariant under the action of the $SU_H(2)$ group, i.e., $\{\!\!\{\Omega_2(\vartheta)T\}\!\!\} = \{\!\!\{T\}\!\!\}$.

^{*)} For simplicity, multiplet structure for the color symmetry is not shown explicitly in this paper.

§3. Field contents in the gauge field theory of $V \times H$ symmetry

In the high energy region, three generations of fermions characterized by the same quantum numbers of the V symmetry and definite chiralities are postulated generically to form the fundamental H triplets as

$$\Psi_h^f(x) = \check{t} \left(\psi_{h1}^f(x), \psi_{h2}^f(x), \psi_{h3}^f(x) \right) \quad (3.1)$$

where $f (= q, u, d; l, \nu, e)$ distinguishes EW multiplets and $h (= L, R)$ refers to chiral components. Specifically, $\Psi_L^f (= \Psi_L^q, \Psi_L^l)$ is the H triplet of EW doublets and $\Psi_R^f (= \Psi_R^u, \Psi_R^d, \Psi_R^\nu, \Psi_R^e)$ is that of EW singlets.

We represent the gauge fields for $SU_c(3)$, $SU_L(2)$ and $U(1)$ subgroups of the V symmetry by $A_\mu^{(3)j}(x)$ ($j = 1, \dots, 8$), $A_\mu^{(2)j}(x)$ ($j = 1, 2, 3$) and $A_\mu^{(1)}(x)$, respectively, and write the gauge coupling constants for $A_\mu^{(k)}(x)$ by g_k ($k = 1, 2, 3$). Corresponding to the generators $\{\check{\tau}_1, \check{\tau}_2, \check{\tau}_3\}$ and $\{\check{D}\}$ of the H symmetry, a triplet $\check{A}_\mu^{(2)j}(x)$ ($j = 1, 2, 3$) and a singlet $\check{A}_\mu^{(1)}(x)$ of the gauge fields are postulated to exist as follows:

$$\check{\tau}_j \leftrightarrow \check{A}_\mu^{(2)j}(x), \quad \check{D} \leftrightarrow \check{A}_\mu^{(1)}(x). \quad (3.2)$$

The gauge fields $\check{A}_\mu^{(k)}(x)$ with coupling constants \check{g}_k have the field strengths

$$\check{F}_{\mu\nu}^{(2)j} = \partial_\mu \check{A}_\nu^{(2)j} - \partial_\nu \check{A}_\mu^{(2)j} + \check{g}_2 \epsilon_{jkl} \check{A}_\mu^{(2)k} \check{A}_\nu^{(2)l}, \quad \check{F}_{\mu\nu}^{(1)} = \partial_\mu \check{A}_\nu^{(1)} - \partial_\nu \check{A}_\mu^{(1)}. \quad (3.3)$$

The gauge fields $A_\mu^{(k)}(x)$ ($k = 1, 2, 3$) and $\check{A}_\mu^{(k)}(x)$ ($k = 1, 2$) of the V and H symmetries belong, respectively, to the singlet representations of the H and V symmetries.

To break properly the H and EM symmetries, two kinds of H multiplets of scalar fields are presumed to exist. For the H symmetry breaking around the high energy scale $\check{\Lambda}$, we introduce three scalar fields $\check{\phi}_j(x)$ which belong to the V singlet and form the fundamental triplet of the H symmetry as

$$\check{\Phi}(x) = \check{t} \left(\check{\phi}_1(x), \check{\phi}_2(x), \check{\phi}_3(x) \right). \quad (3.4)$$

This scalar triplet does not couple with the fermion fields except for the right-handed neutrino triplet $\Psi_R^\nu(x)$. It is this character of $\check{\Phi}(x)$ that prohibits the fermion fields to acquire Dirac masses of the scale $\check{\Lambda}$. Note that a new triplet defined by

$$\tilde{\check{\Phi}}(x) = i\check{\tau}_2 \check{\Phi}^*(x) \quad (3.5)$$

has the same transformation property with $\check{\Phi}(x)$ under the action of $SU_H(2)$ group. We call this triplet as an associated triplet of $\check{\Phi}(x)$.

To form the Yukawa interaction and break it at the scale Λ , a new set of scalar fields belonging to non-trivial multiplets of both groups of EW and H symmetries is required to exist. As a possible choice of such fields, we assume here that three EW

doublets $\Phi_j(x)$ with the EW hypercharge $Y_{EW} = 1$ constitute the H triplet as

$$\begin{aligned}\Phi(x) &= \check{t}(\Phi_1(x), \Phi_2(x), \Phi_3(x)) \\ &= \check{t}\left(\begin{pmatrix} \phi_1^+(x) \\ \phi_1^0(x) \end{pmatrix}, \begin{pmatrix} \phi_2^+(x) \\ \phi_2^0(x) \end{pmatrix}, \begin{pmatrix} \phi_3^+(x) \\ \phi_3^0(x) \end{pmatrix}\right),\end{aligned}\quad (3.6)$$

and define its associated triplet by

$$\begin{aligned}\tilde{\Phi}(x) &= (i\check{\tau}_2)(i\tau_2)\Phi^*(x) = (i\check{\tau}_2)\check{t}(i\tau_2\Phi_1^*(x), i\tau_2\Phi_2^*(x), i\tau_2\Phi_3^*(x)) \\ &= (i\check{\tau}_2)\check{t}\left(\begin{pmatrix} \phi_1^{0*}(x) \\ -\phi_1^-(x) \end{pmatrix}, \begin{pmatrix} \phi_2^{0*}(x) \\ -\phi_2^-(x) \end{pmatrix}, \begin{pmatrix} \phi_3^{0*}(x) \\ -\phi_3^-(x) \end{pmatrix}\right).\end{aligned}\quad (3.7)$$

Both of these triplets have the same transformation properties under the group of H symmetry, $SU_H(2)$, and the EW isospin group, $SU_{EW}(2)$. The H triplets $\tilde{\Phi}(x)$ and $\Phi(x)$ interact, respectively, with the right-handed fermion triplets of EW up- and down-sectors, as in (4.5) and (4.6).

It is postulated that the scalar triplet $\check{\Phi}(x)$ possesses a non-vanishing H hypercharge ($\check{y}_{\check{\Phi}} \neq 0$) and that all other triplets have zero H hypercharge. Evidently, the H-sums $\{\Psi_L^f(x)\}$ and $\{\Phi(x)\}$ are EW doublets and $\{\Psi_R^f(x)\}$ are singlets. The H-sum $\{\check{\Phi}(x)\}$ carrying non-vanishing H hypercharge is invariant under the H and EW symmetries.

Large degrees of freedom in the internal space for the $EW \times H$ symmetry bring about necessarily complications in dynamics of the scalar triplets $\check{\Phi}(x)$ and $\Phi(x)$. There exist various types of identical equations among the EW and H invariants composed of these triplets. To construct the Lagrangian density of the scalar triplets without redundancy, it is necessary to examine all of the invariants and find out relations among them. Analysis on the identical equations is made in Appendix.

§4. Lagrangian density of the $V \times H$ gauge field theory

The Lagrangian density of the theory consists of the parts being dependent and independent of the fermion fields. In construction of the Lagrangian density, it is tacitly understood that the scalar products are taken, without explicitly specifying symbols for the operations, to form the invariants out of V and H multiplets.

The fermionic Lagrangian density is the sum of the kinetic terms including gauge-interactions, $\mathcal{L}_{\Psi A \check{A}}$, and the interaction terms between the fermion and scalar fields, \mathcal{L}_{fs} . The kinetic part $\mathcal{L}_{\Psi A \check{A}}$ consists of the bilinear form of the chiral fermion fields as follows:

$$\mathcal{L}_{\Psi A \check{A}} = \sum_{f,h} \bar{\Psi}_h^f(x) i\gamma^\mu \mathcal{D}_\mu \Psi_h^f(x) \quad (4.1)$$

where \mathcal{D}_μ is the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - i\{V \text{ gauge fields}\}_\mu - i\{H \text{ gauge fields}\}_\mu \quad (4.2)$$

which includes the gauge fields for both of the V and H symmetries. The third term in (4.2) is expressed in terms of the gauge fields $\check{A}_\mu^{(2)j}(x)$ for $SU_H(2)$ group and the generators $\check{\tau}_j$ of the algebra $\check{\mathcal{A}}$ as

$$\{\text{H gauge fields}\}_\mu = \check{g}_2 \check{A}_\mu^{(2)j}(x) \frac{1}{2} \check{\tau}_j \quad (4.3)$$

where the gauge field $\check{A}_\mu^{(1)}(x)$ does not exist, since the zero H hypercharge is assigned to all of the fermionic triplets.

The Lagrangian density \mathcal{L}_{fs} consists of the fermion-scalar interactions of the Yukawa and Majorana types as

$$\mathcal{L}_{\text{fs}} = \sum_{f=u,d,\nu,e} \mathcal{L}_Y^f + \mathcal{L}_M \quad (4.4)$$

where the Yukawa part \mathcal{L}_Y^f includes the fermion triplet of the f -sector ($f = u, d, \nu, e$) and the Majorana part \mathcal{L}_M is composed of the triplet $\Psi_R^\nu(x)$ of right-handed neutrino fields. The $SU_H(2)$ invariance of the quantities $\{\Psi_h^f(x)\}$, $\{\Phi(x)\}$ and $\{\check{\Phi}(x)\}$ brings about ingenious structure for the Yukawa and Majorana interactions.

The density \mathcal{L}_Y^f is constructed by summing up all of the EW \times H invariants consisting of bilinear forms of fermion triplets and the scalar triplets $\Phi(x)$ and $\check{\Phi}(x)$. We find

$$\begin{aligned} \mathcal{L}_Y^f &= Y_{f1} \bar{\Psi}_L^{f'} \check{\Phi} \{\Psi_R^f\} + Y_{f2} \{\bar{\Psi}_L^{f'}\} \check{\Phi} i\check{\tau}_2 \Psi_R^f \\ &+ Y_{f3} \bar{\Psi}_L^{f'} i\tau_2 \{\Phi^*\} \Psi_R^f + Y_{f4} \{\bar{\Psi}_L^{f'}\} i\tau_2 \{\Phi^*\} \{\Psi_R^f\} + \text{h.c.} \end{aligned} \quad (4.5)$$

for the EW up-sectors ($f' = q, f = u$) and ($f' = l, f = \nu$), and

$$\begin{aligned} \mathcal{L}_Y^f &= Y_{f1} \bar{\Psi}_L^{f'} \Phi \{\Psi_R^f\} + Y_{f2} \{\bar{\Psi}_L^{f'}\} \Phi i\tau_2 \Psi_R^f \\ &+ Y_{f3} \bar{\Psi}_L^{f'} \{\Phi\} \Psi_R^f + Y_{f4} \{\bar{\Psi}_L^{f'}\} \{\Phi\} \{\Psi_R^f\} + \text{h.c.} \end{aligned} \quad (4.6)$$

for the EW down-sectors ($f' = q, f = d$) and ($f' = l, f = e$). Each sector includes four unknown complex coupling constants Y_{fi} ($i = 1, \dots, 4$). In these densities, the operation of the H-sum plays an indispensable role to generate $SU_H(2)$ invariants. Without the H-sums which reflects a unique feature of the group $\check{G}(\check{A})$ embodied in (2.10), the Yukawa interaction turns out to be empty in (4.5) and (4.6). The assignment of zero H hypercharge for all of the triplets $\Psi_h^f(x)$ and $\Phi(x)$ is essential in the present formalism.

The Majorana interaction is possible only for the right-handed triplet for neutrino species possessing no SM quantum number. To look for H invariants, it is crucial to remark that the conservation of H hypercharge forbid the quantity $\{\check{\Phi}\}$ to appear and that the element $\check{\tau}_2$ works properly to cancel the H hypercharge effect of the triplet $\check{\Phi}(x)$ owing to the expression for Ω_1 in (2.8) and $\check{\tau}_2 \check{D} = 0$. It is customary to use charge conjugates of neutrino fields to investigate the Majorana interactions. Normally, the charge conjugates of the chiral fermion triplets $\Psi_{L,R}^f$ are defined by

$$\Psi_{L,R}^{fc} = C^{\dagger} \overline{\Psi_{R,L}^f}, \quad \overline{\Psi_{L,R}^{fc}} = \check{\tau}_2 \Psi_{R,L}^f C \quad (4.7)$$

with the charge-conjugation matrix C . The most general Lagrangian density for the Majorana interactions consisting of EW \times H invariants is given in the form

$$\begin{aligned}\mathcal{L}_M &= \check{g}_{M1} {}^{tt}\bar{\Psi}_R^\nu C \check{\tau}_2 \check{\Phi} \{\Psi_R^\nu\} + \check{g}_{M2} {}^t\{\Psi_R^\nu\} C {}^t\check{\Phi} \check{\tau}_2 \Psi_R^\nu + \check{m}_M {}^{tt}\bar{\Psi}_R^\nu C \check{\tau}_2 \Psi_R^\nu + \text{h.c.} \\ &= \check{g}_{M1} \overline{\Psi_L^{\nu c}} \check{\tau}_2 \check{\Phi} \{\Psi_R^\nu\} + \check{g}_{M2} \{\overline{\Psi_L^{\nu c}}\} {}^t\check{\Phi} \check{\tau}_2 \Psi_R^\nu + \check{m}_M \overline{\Psi_L^{\nu c}} \check{\tau}_2 \Psi_R^\nu + \text{h.c.}\end{aligned}\quad (4.8)$$

where $\check{g}_{Mj} (j = 1, 2)$ and \check{m}_M are the Majorana coupling constants and mass.

The Lagrangian density for bosonic fields, which does not include fermion fields, is the sum of the gauge field part and the scalar field part $\mathcal{L}_{\text{scalar}}$. The gauge field density is given by the Lorentz invariants of the field strengths of the V and H gauge fields. The density of the H symmetry, \mathcal{L}_H^G , is constructed as follows:

$$\mathcal{L}_H^G = -\frac{1}{4} \sum_{j=1}^3 \check{F}_{\mu\nu}^{(2)j} \check{F}^{(2)j\mu\nu} - \frac{1}{4} \check{F}_{\mu\nu}^{(1)} \check{F}^{(1)\mu\nu}. \quad (4.9)$$

The Lagrangian density for the scalar fields $\mathcal{L}_{\text{scalar}}$ can be expressed by

$$\mathcal{L}_{\text{scalar}} = (\mathcal{D}^\mu \check{\Phi})^\dagger (\mathcal{D}_\mu \check{\Phi}) + (\mathcal{D}^\mu \Phi)^\dagger (\mathcal{D}_\mu \Phi) - V_T(\check{\Phi}, \Phi) \quad (4.10)$$

where the covariant derivatives for the scalar triplets $\check{\Phi}(x)$ and $\Phi(x)$ are given, respectively, by

$$\mathcal{D}_\mu \check{\Phi} = \left(\partial_\mu - i\check{g}_2 \check{A}_\mu^{(2)j} \frac{1}{2} \check{\tau}_j - i\check{g}_1 \check{A}_\mu^{(1)} \check{y}_{\check{\Phi}} \check{D} \right) \check{\Phi}, \quad (4.11)$$

and

$$\mathcal{D}_\mu \Phi = \left(\partial_\mu - ig_2 A_\mu^{(2)j} \frac{1}{2} \tau_j - ig_1 A_\mu^{(1)} \frac{1}{2} - i\check{g}_2 \check{A}_\mu^{(2)j} \frac{1}{2} \check{\tau}_j \right) \Phi, \quad (4.12)$$

and $V_T(\check{\Phi}, \Phi)$ is the total Higgs potential for both of the triplets $\check{\Phi}(x)$ and $\Phi(x)$. Without loss of generality, we are able to separate the potential $V_T(\check{\Phi}, \Phi)$ into three parts as follows:

$$V_T(\check{\Phi}, \Phi) = V_1(\check{\Phi}) + V_2(\Phi) + V_3(\Phi, \check{\Phi}) \quad (4.13)$$

where $V_1(\check{\Phi})$ and $V_2(\Phi)$ are the potential parts of self-interactions of the triplets $\check{\Phi}(x)$ and $\Phi(x)$, respectively, and $V_3(\Phi, \check{\Phi})$ is the part of their mutual interactions.

To construct explicit forms of the potential parts V_i without double counting, it is necessary to use the identical relations among the invariants of the scalar triplets investigated in Appendix. The potential $V_1(\check{\Phi})$ of the self-interaction is given by

$$\begin{aligned}V_1(\check{\Phi}) &= -\check{m}_1^2 \check{\Phi}^\dagger \check{\Phi} - \check{m}_2^2 \{\check{\Phi}\}^\dagger \{\check{\Phi}\} + \frac{1}{2} \check{\lambda}_1 \left(\check{\Phi}^\dagger \check{\Phi} \right)^2 + \frac{1}{2} \check{\lambda}_2 \left(\{\check{\Phi}\}^\dagger \{\check{\Phi}\} \right)^2 \\ &\quad + \check{\lambda}_3 \left(\check{\Phi}^\dagger \check{\Phi} \right) \left(\{\check{\Phi}\}^\dagger \{\check{\Phi}\} \right)\end{aligned}\quad (4.14)$$

where $\check{\lambda}_1, \check{\lambda}_2$ and $\check{\lambda}_3$ are positive-definite coupling constants. Using this density, we analyze the breakdown of the H symmetry, preserving the EW symmetry, around the scale $\check{\Lambda}$. Since the triplet Φ carries the EW hypercharge $Y_{\text{EW}} = 1$, the quantity

$\Phi^\dagger \tilde{\Phi}$ in (A.10) is not the EW invariant. It appears in the scalar potential only as the product with its Hermite conjugate. In consequence, we obtain

$$\begin{aligned} V_2(\Phi) = & -m_1^2 \Phi^\dagger \Phi - m_2^2 \{\{\Phi\}\}^\dagger \{\{\Phi\}\} + \frac{1}{2} \bar{\lambda}_1 (\Phi^\dagger \Phi)^2 + \frac{1}{2} \bar{\lambda}_2 (\{\{\Phi\}\}^\dagger \{\{\Phi\}\})^2 \\ & + \bar{\lambda}_3 (\Phi^\dagger \Phi) (\{\{\Phi\}\}^\dagger \{\{\Phi\}\}) + \bar{\lambda}_4 |\Phi^\dagger \{\{\Phi\}\}|^2 + \bar{\lambda}_5 \Phi^\dagger i\tau_2 {}^t\tilde{\Phi}^* {}^t\Phi i\tau_2 \Phi \\ & + \tilde{\lambda}_1 |\Phi^\dagger \tilde{\Phi}|^2 + \tilde{\lambda}_2 |\tilde{\Phi}^\dagger \{\{\Phi\}\}|^2 + \tilde{\lambda}_3 |\tilde{\Phi}^\dagger (i\tau_2) \{\{\Phi^*\}\}|^2 \end{aligned} \quad (4.15)$$

where, in the term with the coupling constant $\bar{\lambda}_5$, the H scalar products are taken between the first Φ^\dagger and the last Φ and between the middle ${}^t\tilde{\Phi}^*$ and ${}^t\Phi$.

Finally, the potential for mutual-interactions between the triplets $\Phi(x)$ and $\check{\Phi}(x)$ is proved to take the form:

$$\begin{aligned} V_3(\Phi, \check{\Phi}) = & \dot{\lambda}_1 (\Phi^\dagger \Phi) (\check{\Phi}^\dagger \check{\Phi}) + \dot{\lambda}_2 (\Phi^\dagger \Phi) (\{\{\check{\Phi}\}\}^\dagger \{\{\check{\Phi}\}\}) + \dot{\lambda}_3 (\{\{\Phi\}\}^\dagger \{\{\Phi\}\}) (\check{\Phi}^\dagger \check{\Phi}) \\ & + \dot{\lambda}_4 (\{\{\Phi\}\}^\dagger \{\{\Phi\}\}) (\{\{\check{\Phi}\}\}^\dagger \{\{\check{\Phi}\}\}) + \dot{\lambda}_5 (\tilde{\Phi}^\dagger \tilde{\Phi}) (\check{\Phi}^\dagger \check{\Phi}) + \dot{\lambda}_6 |\Phi^\dagger (I - \check{D}) \check{\Phi}|^2 + \dot{\lambda}_7 |\Phi^\dagger \check{D} \check{\Phi}|^2. \end{aligned} \quad (4.16)$$

§5. Remark on a compound mechanism of symmetry breakdowns

At present, there exists no established theory which can describe consistently multi-stage-breakdowns of symmetries with vastly-different energy scales. We formulate, here, a compound mechanism of symmetry breakdowns that can lead, without creating unphysical modes of massless particles and tachyons, to an effective theory with lesser numbers of unknown parameters as possible for low energy phenomenology. For its purpose, let us postulate first that the triplet $\check{\Phi}(x)$ ($\Phi(x)$) takes a stationary reference state^{*)} specified by a single parameter. And then, we decompose the triplet around the reference state and impose necessary modifications so as to forbid unphysical modes to appear.

Around the high energy scale $\check{\Lambda}$, the reference state of the triplet $\check{\Phi}$ is derived by neglecting effects of the triplet $\Phi(x)$. Let us assume that the reference state takes the form

$$\langle \check{\Phi} \rangle = {}^t(0, 0, \check{v}) = -\sqrt{\frac{2}{3}} \check{v} |2\rangle + \frac{1}{\sqrt{3}} \check{v} |3\rangle \quad (5.1)$$

where $\check{v} \approx \check{\Lambda}$. Following the standard procedure, we calculate a stationary point of $V_1(\langle \check{\Phi} \rangle) = V_1(\check{v})$ by differentiating it with respect to \check{v} and obtain

$$\check{v}^2 = \frac{\check{m}_1^2 + \check{m}_2^2}{\check{\lambda}_1 + \check{\lambda}_2 + 2\check{\lambda}_3}. \quad (5.2)$$

It is natural to consider that the breakdown of the H symmetry might give influence on the symmetry breakdown in low energy region around the scale Λ .

^{*)} In this trial framework of the symmetry breakdown, we abstain from using the concept of the “vacuum” which describes the stable potential minimum of the quantum system of the scalar multiplets.

We look for a reference state for the triplet $\Phi(x)$ by examining a local symmetry-violating stationary-point of the potential $V_2(\Phi) + V_3(\Phi, \check{v})$. To deduce a compact effective theory for low energy, let us assume, here again, the scalar triplet $\Phi(x)$ takes the reference state in the simplest form

$$\langle \Phi \rangle = \check{v} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_0 \end{pmatrix} \right) = -\sqrt{\frac{2}{3}} \begin{pmatrix} 0 \\ v_0 \end{pmatrix} |2\rangle + \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ v_0 \end{pmatrix} |3\rangle \quad (5.3)$$

where v_0 is a parameter specifying a local symmetry-violating stationary-point. Taking derivative of $V_2(\langle \Phi \rangle) + V_3(\langle \Phi \rangle, \check{v})$ with respect to v_0 , we obtain

$$v_0^2 = \frac{m_1^2 + m_2^2 - (\dot{\lambda}_1 + \dot{\lambda}_2 + \dot{\lambda}_3 + \dot{\lambda}_4 + \frac{4}{9}\dot{\lambda}_6 + \frac{1}{9}\dot{\lambda}_7)\check{v}^2}{\bar{\lambda}_1 + \bar{\lambda}_2 + 2\bar{\lambda}_3 + 2\bar{\lambda}_4 + \frac{4}{3}\widetilde{\lambda}_3}. \quad (5.4)$$

Evidently, the simplest choices of the reference states in (5.1) and (5.3) break only partially the H symmetry. As shown explicitly in the following sections, the remaining symmetry causes massless Nambu-Goldstone (NG) particles to appear. Fortunately, there is a mechanism to circumvent such difficulties by utilizing the specific property of the vector of the reference state that can be expressed uniquely in terms of linear combination of the eigenvectors $|2\rangle$ and $|3\rangle$, as shown in the right-hand sides of (5.1) and (5.3). This feature enables us to decompose the triplet around the reference state so that one component field can be shared in both coefficients of $|2\rangle$ and $|3\rangle$ by accompanying an adjustable mixing parameter. Such decompositions of the triplets $\check{\Phi}(x)$ and $\Phi(x)$ are, respectively, given in (6.2) and (7.4) below. It is this facility of sharing the component field with mixing parameter that acts to break the residual symmetry and suppress the appearance of massless scalar fields in the phases of broken symmetries.

Substitution of the decompositions of the scalar triplets in the broken phase into the Lagrangian density produces, necessarily, anomalous unphysical terms depending linearly on some of the component scalar fields. It is possible to set the coefficients of such harmful terms to disappear by adjusting finite values appropriately for the mixing parameters in the decompositions in (6.2) and (7.7). Consequently, this mechanism which affords non-vanishing mixing parameters breaks the residual symmetry and enables the would-be NG modes to acquire finite masses.

In the low energy symmetry breakdown, however, this mixing mechanism of component fields is not enough to forbid the decomposition of the triplet $\Phi(x)$ around the reference state in (7.4) to have a component field with an imaginary mass. Another device is necessary to complete the compound mechanism of the symmetry breakdowns. To suppress a tachyonic mode to appear, we have to utilize a freedom of rescaling of the parameter v_0 as made in (7.3).

§6. Symmetry breakdown at high energy scale $\check{\Lambda}$

In the broken phase of the H symmetry around the scale $\check{\Lambda} = \check{v}$, the scalar triplet field $\check{\Phi}(x)$ can be decomposed into

$$\check{\Phi}(x) = \Omega(\check{v}(x)) \check{\Phi}_0(x) \quad (6.1)$$

where $\Omega(\check{\vartheta}(x))$ is the unitary group element of the H symmetry including the local fields $\check{\vartheta}_j(x)$ ($j = 0, 1, 2, 3$) destined to be gauged away. The gauge-fixed part $\check{\Phi}_0(x)$ is assumed to take the following form as

$$\check{\Phi}_0(x) = \frac{1}{\sqrt{2}}\check{\xi}_1(x)|1\rangle + \left(\frac{1}{\sqrt{2}}\check{\alpha}\check{\xi}_2(x) - \sqrt{\frac{2}{3}}\check{v}\right)|2\rangle + \left(\frac{1}{\sqrt{2}}\check{\beta}\check{\xi}_2(x) + \frac{1}{\sqrt{3}}\check{v}\right)|3\rangle \quad (6.2)$$

in which $\check{\xi}_1(x)$ and $\check{\xi}_2(x)$ are real scalar fields, and $\check{\alpha} = \cos \check{\theta}$ and $\check{\beta} = \sin \check{\theta}$ are parameters for mixing. With this gauge-fixing of the scalar triplet $\check{\Phi}(x)$, the gauge fields are metamorphosed to the massive vector fields through the local gauge transformation $\check{A}_\mu(x) \rightarrow \check{A}'_\mu(x)$ defined by

$$\mathcal{D}_\mu(\check{A}') = \Omega^{-1}(\check{\vartheta}(x))\mathcal{D}_\mu(\check{A})\Omega(\check{\vartheta}(x)) \quad (6.3)$$

where $\mathcal{D}_\mu(\check{A})$ is the covariant derivative for the scalar triplet $\check{\Phi}(x)$ in (4.11).

The gauge-fixing of the H symmetry imposed on the scalar triplet $\check{\Phi}(x)$ gives necessarily influences on all other fields. Namely, it necessitates inevitably to adjust the same unitary factor $\Omega(\check{\vartheta}(x))$ for the scalar and fermion triplets, $\check{\Phi}(x)$ and $\check{\Psi}_h^f(x)$, leaving the gauge-fixed parts $\check{\underline{\Phi}}(x)$ and $\check{\underline{\Psi}}_h^f(x)$ as follows:

$$\check{\Phi}(x) = \Omega(\check{\vartheta}(x))\check{\underline{\Phi}}(x), \quad \check{\Psi}_h^f(x) = \Omega(\check{\vartheta}(x))\check{\underline{\Psi}}_h^f(x). \quad (6.4)$$

Substitution of the decomposition (6.1) and (6.2) into the potential $V_1(\check{\Phi})$ in (4.14) leads to

$$\begin{aligned} V_1(\check{\Phi}_0) = & \check{v} \left\{ \sqrt{\frac{2}{3}} \left[\check{m}_1^2 - (\check{\lambda}_1 + \check{\lambda}_3)\check{v}^2 \right] (\sqrt{2}\check{\alpha} - \check{\beta}) - \sqrt{6} \left[\check{m}_2^2 - (\check{\lambda}_2 + \check{\lambda}_3)\check{v}^2 \right] \check{\beta} \right\} \check{\xi}_2 \\ & + \frac{1}{2}[-\check{m}_1^2 + (\check{\lambda}_1 + \check{\lambda}_3)\check{v}^2] (\check{\xi}_1^2 + \check{\xi}_2^2) \\ & + \frac{1}{2} \left\{ -3\check{\beta}^2 \check{m}_2^2 + \left[\frac{2}{3}\check{\lambda}_1(\sqrt{2}\check{\alpha} - \check{\beta})^2 + 3(3\check{\lambda}_2 + \check{\lambda}_3)\check{\beta}^2 - 4\check{\lambda}_3(\sqrt{2}\check{\alpha} - \check{\beta})\check{\beta} \right] \check{v}^2 \right\} \check{\xi}_2^2 \\ & - \sqrt{\frac{2}{3}} \left[\check{\lambda}_1(\sqrt{2}\check{\alpha} - \check{\beta}) - \frac{3}{2}\check{\lambda}_3\check{\beta} \right] \check{v} (\check{\xi}_1^2 + \check{\xi}_2^2) \check{\xi}_2 + \sqrt{\frac{3}{2}} \left[3\check{\lambda}_2\check{\beta} - \check{\lambda}_3(\sqrt{2}\check{\alpha} - \check{\beta}) \right] \check{\beta}^2 \check{v} \check{\xi}_2^3 \\ & + \frac{1}{8}\check{\lambda}_1 (\check{\xi}_1^2 + \check{\xi}_2^2)^2 + \frac{9}{8}\check{\lambda}_2\check{\beta}^4 \check{\xi}_2^4 + \frac{3}{4}\check{\lambda}_3\check{\beta}^2 (\check{\xi}_1^2 + \check{\xi}_2^2) \check{\xi}_2^2 + V_1(\check{v}). \end{aligned} \quad (6.5)$$

The first term of this reduced potential is linear with respect to the field $\check{\xi}_2(x)$. Postulating the coefficient of this unphysical term to vanish, we obtain the condition which fixes the parameter $\check{\theta}$ as follows:

$$\tan \check{\theta} = \sqrt{2} \frac{\check{m}_1^2 - (\check{\lambda}_1 + \check{\lambda}_3)\check{v}^2}{\check{m}_1^2 + 3\check{m}_2^2 - (\check{\lambda}_1 + 3\check{\lambda}_2 + 4\check{\lambda}_3)\check{v}^2}. \quad (6.6)$$

The masses of the real scalar fields $\check{\xi}_1(x)$ and $\check{\xi}_2(x)$ are calculated to be

$$m_{\check{\xi}_1}^2 = -\check{m}_1^2 + (\check{\lambda}_1 + \check{\lambda}_3)\check{v}^2 \propto \sin \check{\theta} \quad (6.7)$$

and

$$m_{\xi_2}^2 = m_{\xi_1}^2 - 3\check{m}_2^2 \sin^2 \check{\theta} + \frac{1}{3} \left[4\check{\lambda}_1 \cos^2 \check{\theta} - 2\sqrt{2}(\check{\lambda}_1 + 3\check{\lambda}_3) \sin 2\check{\theta} + (2\check{\lambda}_1 + 27\check{\lambda}_2 + 21\check{\lambda}_3) \sin^2 \check{\theta} \right] \check{v}^2. \quad (6.8)$$

The mass of the field $\check{\xi}_1(x)$ is proportional to $\sin \check{\theta}$. If $\check{\theta} = 0$, the field $\check{\xi}_1(x)$ remain necessarily in a massless mode. Therefore, the sharing of the component field $\check{\xi}_2(x)$ in both of the coefficients of $|2\rangle$ and $|3\rangle$ in (6.2) is indispensable to break the residual symmetry and set free the fields $\check{\xi}_1(x)$ from the NG theorem.

To derive configurations of massive vector fields $\check{A}'_\mu(x)$ which are related with the gauge fields $\check{A}_\mu(x)$ by the transformation in (6.3), it is sufficient to calculate the action of the covariant derivative on the reference state in (5.1) as follows:

$$\begin{aligned} \mathcal{D}_\mu(\check{A}')\langle\check{\Phi}\rangle &= \left(\partial_\mu - i\check{g}_2 \check{A}_\mu^{(2)'} j \frac{1}{2} \check{\tau}_j - i\check{g}_1 \check{A}_\mu^{(1)'} \check{y}_{\check{\Phi}} \check{D} \right) \langle\check{\Phi}\rangle \\ &= i\frac{1}{2} \begin{pmatrix} \sqrt{2}M_{\check{W}}\check{W}_\mu - M_{\check{Y}}\check{Y}_\mu \\ -\sqrt{2}M_{\check{W}}\check{W}_\mu - M_{\check{Y}}\check{Y}_\mu \\ \sqrt{2}M_{\check{Z}}\check{Z}_\mu \end{pmatrix} \end{aligned} \quad (6.9)$$

where the vector fields $\check{W}_\mu(x)$, $\check{Y}_\mu(x)$ and $\check{Z}_\mu(x)$ are expressed, tentatively, by

$$\check{W}_\mu = \frac{\check{A}_\mu^{(2)'} - i\check{A}_\mu^{(2)'2}}{\sqrt{2}}, \quad \check{Y}_\mu = \frac{\check{g}_2 \check{A}_\mu^{(2)'}3 + 2\check{g}_1 \check{y}_{\check{\Phi}} \check{A}_\mu^{(1)'}}{\sqrt{\check{g}_2^2 + 4\check{g}_1^2 \check{y}_{\check{\Phi}}^2}}, \quad \check{Z}_\mu = \frac{\check{g}_2 \check{A}_\mu^{(2)'}3 - \check{g}_1 \check{y}_{\check{\Phi}} \check{A}_\mu^{(1)'}}{\sqrt{\check{g}_2^2 + \check{g}_1^2 \check{y}_{\check{\Phi}}^2}}, \quad (6.10)$$

and their masses are given, respectively, by $M_{\check{W}}^2 = \frac{1}{3}\check{g}^2\check{v}^2$, $M_{\check{Y}}^2 = \frac{1}{9}(\check{g}_2^2 + 4\check{g}_1^2 \check{y}_{\check{\Phi}}^2)\check{v}^2$ and $M_{\check{Z}}^2 = \frac{2}{9}(\check{g}_2^2 + \check{g}_1^2 \check{y}_{\check{\Phi}}^2)\check{v}^2$.

So far, no restriction is assumed to exist among the gauge coupling constants \check{g}_2 and \check{g}_1 and the H hypercharge $\check{y}_{\check{\Phi}}$. Consistency of the theory, however, requires an additional relation among them. Namely, calculation of kinetic terms of the vector fields $\check{Y}_\mu(x)$ and $\check{Z}_\mu(x)$ by substituting the inverse relations of (6.10) into the Lagrangian density $\mathcal{L}_H^G(\check{A}_\mu) = \mathcal{L}_H^G(\check{A}'_\mu)$ in (4.9) proves that, there arise unphysical kinetic terms such as $\partial_\mu \check{Y}_\nu \partial^\mu \check{Z}^\nu$, unless the relation

$$\check{g}_2^2 = 2\check{g}_1^2 \check{y}_{\check{\Phi}}^2 \quad (6.11)$$

holds. This condition allows to relate the fields $\check{Y}_\mu(x)$ and $\check{Z}_\mu(x)$ with the fields $\check{A}_\mu^{(2)'}(x)$ and $\check{A}_\mu^{(1)'}(x)$ by the orthogonal transformation

$$\check{Y}_\mu = \frac{1}{\sqrt{3}}\check{A}_\mu^{(2)'}3 + \sqrt{\frac{2}{3}}\check{A}_\mu^{(1)'}, \quad \check{Z}_\mu = \sqrt{\frac{2}{3}}\check{A}_\mu^{(2)'}3 - \frac{1}{\sqrt{3}}\check{A}_\mu^{(1)'} \quad (6.12)$$

and results in the degenerate masses, i.e., $M_{\check{Y}}^2 = M_{\check{Z}}^2 = M_{\check{W}}^2 = \frac{1}{3}\check{g}_2^2\check{v}^2$.

In this way, the compound mechanism of symmetry breakdown at the scale $\check{\Lambda}$ succeeds to transform all of the massless gauge and scalar fields into the massive vector and scalar fields. To accomplish these results, it is necessary to assume that the triplet $\check{\Phi}(x)$ possesses a non-vanishing H hypercharge satisfying the relation in (6.11), and that the mixing angle of its decomposition in (6.2) takes the definite value obeying (6.6).

Through the symmetry-breaking at the scale $\check{\Lambda}$, the freedoms of the triplet $\check{\Phi}(x)$ are transferred to the gauge fields. A balance sheet of transferring of the field degrees of freedom at this phase transition is schematically summarized as follows:

$$\left\{ \begin{array}{ll} 4 \text{ gauge fields } \check{A}_\mu^{(2)i}(x) (i = 1, 2, 3) \text{ and } \check{A}_\mu^{(1)}(x) & : 4 \times 2 \\ 3 \text{ massless complex scalar fields } \check{\phi}_i(x) (i = 1, 2, 3) & : 3 \times 2 \end{array} \right\} \quad \Downarrow \quad (6.13)$$

$$\left\{ \begin{array}{ll} 1 \text{ massive complex vector fields } \check{W}_\mu(x) & : 2 \times 3 \\ 2 \text{ massive real vector fields } \check{Y}_\mu(x) \text{ and } \check{Z}_\mu(x) & : 2 \times 3 \\ 2 \text{ massive real scalar fields } \check{\xi}_i(x) (i = 1, 2) & : 2 \times 1 \end{array} \right\}.$$

Here the number of modes of independent fields is preserved as $4 \times 2 + 3 \times 2 = 14 = 2 \times 3 + 2 \times 3 + 2 \times 1$ before and after the phase transition.

Substitution of the decomposition of $\check{\Phi}_0$ in (6.2) into (4.8) leads to the effective Lagrangian density of neutrino species as

$$\mathcal{L}_M \rightarrow \mathcal{L}_M^M = \overline{\Psi}_L^{\nu c} \check{\mathcal{M}}_\nu \Psi_R^\nu + \text{h.c.} + \dots \quad (6.14)$$

where the ellipsis stands for the interactions of neutrinos with the scalar fields $\check{\xi}_i(x)$, and $\check{\mathcal{M}}_\nu$ is the Majorana mass matrix

$$\check{\mathcal{M}}_\nu = \frac{1}{\sqrt{3}} B_{\nu 1} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{3}} B_{\nu 2} \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} + C_\nu \check{\tau}_2 \quad (6.15)$$

in which the coefficients are given by $B_{\nu 1} = \check{g}_{M1} \check{v}$, $B_{\nu 2} = \check{g}_{M2} \check{v}$ and $C_\nu = \check{m}_M$. Note that this mass matrix is characterized by the same number of parameters with the coupling constants in the Majorana interaction.

§7. Symmetry breakdown at low energy scale Λ

In the energy region below and close the scale $\check{\Lambda}$, it should be the reduced Lagrangian density that describes dynamics of the triplet $\check{\Phi}(x)$ which loses the unitary phase factor of the H symmetry in (6.4). Subsequently, to go down to the low energy region around the scale Λ , effects of the renormalization group must be taken into account for all of the physical quantities. In particular, all coupling constants run down to the scale Λ . For the sake of simplicity, the same symbols are used here for the quantities including all these effects. To make analysis on the EW symmetry

breakdown around the scale Λ , we have to examine the potential $V_2(\underline{\Phi})$ and the part

$$\begin{aligned} V_3(\underline{\Phi}, \check{v}) &= (\dot{\lambda}_1 + \dot{\lambda}_2) \check{v}^2 \underline{\Phi}^\dagger \underline{\Phi} + (\dot{\lambda}_3 + \dot{\lambda}_4) \check{v}^2 \{\{\underline{\Phi}\}\}^\dagger \{\{\underline{\Phi}\}\} \\ &\quad + \frac{2}{3} \dot{\lambda}_5 \check{v}^2 |\langle \underline{\Phi} | 1 \rangle|^2 + \frac{2}{3} \dot{\lambda}_6 \check{v}^2 |\langle \underline{\Phi} | 2 \rangle|^2 + \frac{1}{3} \dot{\lambda}_7 \check{v}^2 |\langle \underline{\Phi} | 3 \rangle|^2 \end{aligned} \quad (7.1)$$

which reflects the influence of the H symmetry breakdown.

To investigate behaviors of the triplets $\underline{\Phi}(x)$ and $\underline{\Psi}(x)$ in the broken phase of the EW symmetry, it is necessary further to separate the unitary gauge factor as

$$\underline{\Phi}(x) = \Omega_{\text{EW}}(\vartheta(x)) \Phi_0(x), \quad \underline{\Psi}_h^f(x) = \Omega_{\text{EW}}(\vartheta(x)) \Psi_{h0}^f(x) \quad (7.2)$$

where $\Omega_{\text{EW}}(\vartheta(x))$ is a group element of the EW symmetry, which includes local component fields to be gauged away. As noticed in §5, the decomposition of the triplet Φ_0 around the state in (5.3) has a danger to bring about component fields of zero and imaginary masses. In order to suppress the tachyonic mode to appear, we rescale the parameter v_0 into a new value by

$$v^2 = Z v_0^2 \quad (Z > 1) \quad (7.3)$$

and introduce the following decomposition of the $\Phi_0(x)$ as

$$\Phi_0(x) = \begin{pmatrix} \zeta_1^+(x) \\ \zeta_1^0(x) \end{pmatrix} |1\rangle + \begin{pmatrix} \zeta_2^+(x) \\ \zeta_2^0(x) - \sqrt{\frac{2}{3}}v \end{pmatrix} |2\rangle + \begin{pmatrix} 0 \\ \zeta_3^0(x) + \frac{1}{\sqrt{3}}v \end{pmatrix} |3\rangle \quad (7.4)$$

where $\zeta_i^+(x)$ ($i = 1, 2$) and $\zeta_i^0(x)$ ($i = 1, 2, 3$) are complex scalar component fields. As confirmed below, the condition $Z > 1$ is required to exclude a tachyonic mode.

The representation of $\Phi(x)$ in (6.4) ((7.2) and (7.4)) must be substituted into the sum of the potentials $V_2(\Phi)$ in (4.15) and $V_3(\Phi, \check{v})$ in (7.1). Up to the second order with respect to the component scalar fields, we obtain

$$\begin{aligned} V_2(\zeta) + V_3(\zeta, \check{v}) &= -\sqrt{\frac{2}{3}}v [-m_1'^2 + (\bar{\lambda}_1 + \bar{\lambda}_3 + \bar{\lambda}_4 + \tilde{\lambda}_3)v^2 + \frac{2}{3}\dot{\lambda}_6\check{v}^2](\zeta_2^{0*} + \zeta_2^0) \\ &\quad + \sqrt{\frac{1}{3}}v [-m_1'^2 - 3m_2'^2 + (\bar{\lambda}_1 + 3\bar{\lambda}_2 + 4\bar{\lambda}_3 + 4\bar{\lambda}_4 + 2\tilde{\lambda}_3)v^2 + \frac{1}{3}\dot{\lambda}_7\check{v}^2](\zeta_3^{0*} + \zeta_3^0) \\ &\quad + [-m_1'^2 + (\bar{\lambda}_1 + \bar{\lambda}_3 + \tilde{\lambda}_2)v^2](|\zeta_1^+|^2 + |\zeta_2^+|^2) - 2\bar{\lambda}_5 v_0^2 (|\zeta_1^+|^2 + \frac{1}{3}|\zeta_2^+|^2) \\ &\quad + \frac{8}{3}\tilde{\lambda}_1 v^2 |\zeta_1^+|^2 + [-m_1'^2 + (\bar{\lambda}_1 + \bar{\lambda}_3 + \bar{\lambda}_4 + \tilde{\lambda}_3)v^2](|\zeta_1^0|^2 + |\zeta_2^0|^2) \\ &\quad + \frac{2}{3}\dot{\lambda}_5 \check{v}^2 (|\zeta_1^+|^2 + |\zeta_1^0|^2) + \frac{2}{3}\dot{\lambda}_6 \check{v}^2 (|\zeta_2^+|^2 + |\zeta_2^0|^2) + \frac{1}{3}\dot{\lambda}_7 \check{v}^2 |\zeta_3^0|^2 \\ &\quad + \left[-m_1'^2 - 3m_2'^2 + (\bar{\lambda}_1 + 3\bar{\lambda}_2 + 4\bar{\lambda}_3 + 4\bar{\lambda}_4 + 2\tilde{\lambda}_3)v^2 \right] |\zeta_3^0|^2 \\ &\quad + \frac{1}{3}\bar{\lambda}_1 v^2 (\zeta_2^{0*} + \zeta_2^0)^2 + \left(\frac{1}{6}\bar{\lambda}_1 + \frac{3}{2}\bar{\lambda}_2 + \bar{\lambda}_3 + \bar{\lambda}_4 \right) v^2 (\zeta_3^{0*} + \zeta_3^0)^2 \\ &\quad - \sqrt{2} \left(\frac{1}{3}\bar{\lambda}_1 + \bar{\lambda}_3 + \bar{\lambda}_4 + \tilde{\lambda}_3 \right) v^2 (\zeta_2^{0*} + \zeta_2^0)(\zeta_3^{0*} + \zeta_3^0) + \dots \end{aligned} \quad (7.5)$$

where

$$m_1'^2 = m_1^2 - (\check{\lambda}_1 + \check{\lambda}_2)\check{v}^2, \quad m_2'^2 = m_2^2 - (\check{\lambda}_3 + \check{\lambda}_4)\check{v}^2. \quad (7.6)$$

The underlined parts are unphysical harmful terms depending linearly on $\zeta_2^{0*}(x) + \zeta_2^0(x)$ and $\zeta_3^{0*}(x) + \zeta_3^0(x)$. To eliminate such terms, let us reexpress the component fields $\zeta_2^0(x)$ and $\zeta_3^0(x)$ by

$$\zeta_2^0(x) = \frac{1}{\sqrt{2}}[\eta_1(x) \cos \theta + i\eta_2(x)], \quad \zeta_3^0(x) = \frac{1}{\sqrt{2}}[\eta_1(x) \sin \theta + i\eta_3(x)] \quad (7.7)$$

in terms of new real fields $\eta_i(x)$ ($i = 1, 2, 3$) and a mixing angle θ . Then the underlined terms in (7.5) are absorbed into a single term depending linearly on the new field $\eta_1(x)$. Requiring such a harmful term to vanish, we find the constraint which fixes the mixing angle θ as

$$\tan \theta = \sqrt{2} \frac{m_1'^2 - (\bar{\lambda}_1 + \bar{\lambda}_3 + \bar{\lambda}_4 + \tilde{\lambda}_3)v^2 + \frac{2}{3}\dot{\lambda}_6\check{v}^2}{m_1'^2 + 3m_2'^2 - (\bar{\lambda}_1 + 3\bar{\lambda}_2 + 4\bar{\lambda}_3 + 4\bar{\lambda}_4 + 2\tilde{\lambda}_3)v^2 + \frac{1}{3}\dot{\lambda}_7\check{v}^2}. \quad (7.8)$$

Consequently, the second stage breakdown of the symmetry induces the three complex fields $\zeta_1^+(x)$, $\zeta_2^+(x)$ and $\zeta_1^0(x)$, and the three real fields $\eta_i(x)$ ($i = 1, 2, 3$). The complex fields possess the masses

$$\begin{aligned} m_{\zeta_1^+}^2 &= -m_1'^2 + (\bar{\lambda}_1 + \bar{\lambda}_3 - 2\bar{\lambda}_5 + \frac{8}{3}\tilde{\lambda}_1 + \tilde{\lambda}_2)v^2 + \frac{2}{3}\dot{\lambda}_5\check{v}^2, \\ m_{\zeta_2^+}^2 &= -m_1'^2 + (\bar{\lambda}_1 + \bar{\lambda}_3 - \frac{2}{3}\bar{\lambda}_5 + \tilde{\lambda}_2)v^2 + \frac{2}{3}\dot{\lambda}_6\check{v}^2, \\ m_{\zeta_1^0}^2 &= -m_1'^2 + (\bar{\lambda}_1 + \bar{\lambda}_3 + \bar{\lambda}_4 + \tilde{\lambda}_3)v^2 + \frac{2}{3}\dot{\lambda}_5\check{v}^2. \end{aligned} \quad (7.9)$$

The masses of the three real fields are calculated to be

$$\begin{aligned} m_{\eta_1}^2 &= m_{\eta_2}^2 \cos^2 \theta + m_{\eta_3}^2 \sin^2 \theta \\ &\quad + 4 \left[\frac{1}{3}\bar{\lambda}_1 \cos^2 \theta + \left(\frac{1}{6}\bar{\lambda}_1 + \frac{3}{2}\bar{\lambda}_2 + \bar{\lambda}_3 + \bar{\lambda}_4 \right) \sin^2 \theta \right. \\ &\quad \left. - \sqrt{2} \left(\frac{1}{3}\bar{\lambda}_1 + \bar{\lambda}_3 + \bar{\lambda}_4 + \tilde{\lambda}_3 \right) \cos \theta \sin \theta \right] v^2 \\ &\quad + \left(\frac{2}{3}\dot{\lambda}_6 \cos^2 \theta + \frac{1}{3}\dot{\lambda}_7 \sin^2 \theta \right) \check{v}^2, \\ m_{\eta_2}^2 &= -m_1'^2 + (\bar{\lambda}_1 + \bar{\lambda}_3 + \bar{\lambda}_4 + \tilde{\lambda}_3)v^2 + \frac{2}{3}\dot{\lambda}_6\check{v}^2 \propto \sin \theta, \\ m_{\eta_3}^2 &= -m_1'^2 - 3m_2'^2 + (\bar{\lambda}_1 + 3\bar{\lambda}_2 + 4\bar{\lambda}_3 + 4\bar{\lambda}_4 + 2\tilde{\lambda}_3)v^2 + \frac{1}{3}\dot{\lambda}_7\check{v}^2. \end{aligned} \quad (7.10)$$

Note that $m_{\eta_2}^2$ is proportional to $\sin \theta$. This means that the real scalar field $\eta_2(x)$ corresponds to the would-be NG boson, since it becomes massless if $\theta = 0$.

These expressions for the squared masses of six boson fields include more than six unknown adjustable coupling constants. Therefore, it seems possible to make all of the squared masses to be positive by choosing properly the values of the coupling constants. This is, however, not the case. The expression for v_0^2 in (5.4) results readily in the identity

$$2m_{\eta_2}^2 + m_{\eta_3}^2 = (Z - 1)(3\bar{\lambda}_1 + 3\bar{\lambda}_2 + 6\bar{\lambda}_3 + 6\bar{\lambda}_4 + 4\tilde{\lambda}_3)v^2 \quad (7.11)$$

between $m_{\eta_2}^2$ and $m_{\eta_3}^2$. Therefore, if $Z = 1$, either $m_{\eta_2}^2$ or $m_{\eta_3}^2$ must be negative. To exclude such a tachyonic mode, it is necessary to rescale the value v_0 to v and impose the condition $Z > 1$.

Let us indicate the covariant derivative for the scalar triplet $\Phi(x)$ in (4.12) by $\mathcal{D}_\mu(A, \check{A})$ to show its dependence on the gauge fields. Then the gauge-fixings for the H and EW symmetries transform it as follows:

$$\mathcal{D}_\mu(A', \check{A}') = \Omega^{-1}(\check{\vartheta}(x)) \Omega_{\text{EW}}^{-1}(\vartheta(x)) \mathcal{D}_\mu(A, \check{A}) \Omega_{\text{EW}}(\vartheta(x)) \Omega(\check{\vartheta}(x)) \quad (7.12)$$

where $A'_\mu(x)$ is the vector fields in the phase of broken symmetries. To derive configurations of the transformed vector fields $A'_\mu(x)$ and examine their interaction with the scalar component fields, it is necessary to investigate the action of this covariant derivative on the scalar triplet $\Phi_0(x)$ in (7.4). Note that, in such calculation, effects of the super-massive vector fields $\check{W}_\mu(x)$, $\check{Y}_\mu(x)$ and $\check{Z}_\mu(x)$ can safely be ignored. To determine the configurations of the vector fields $A'_\mu(x)$, it is sufficient to calculate the action of the covariant derivative $\mathcal{D}_\mu(A', 0)$ on the triplet $\Phi_0(x)$. Consequently, we can reproduce all of the results of the Weinberg-Salam theory and determine interactions of the scalar fields $\zeta_i^+(x)$ ($i = 1, 2$), $\zeta_1^0(x)$ and $\eta_i(x)$ ($i = 1, 2, 3$) with the electromagnetic field $A_\mu(x)$ and the weak boson fields $W_\mu(x)$ and $Z_\mu(x)$ which possess, respectively, the masses $M_W^2 = \frac{1}{2}g_2^2 v^2$ and $M_Z^2 = \frac{1}{2}(g_2^2 + g_1^2)v^2$.

In total, the triplet $\Phi(x)$ is proved to break the H and EW symmetries at the scale Λ and to create the weak boson fields as well as the electromagnetic field from the EW gauge fields without leaving any massless scalar fields at all. Here, the transferring of freedoms of the scalar fields to the gauge fields is traced as follows:

$$\left\{ \begin{array}{ll} 2 \text{ gauge fields } A_\mu^{(2)1}(x) \text{ and } A_\mu^{(2)2}(x) & : 2 \times 2 \\ 3 \text{ massless complex scalar fields } \phi_i^+(x) (i = 1, 2, 3) & : 3 \times 2 \end{array} \right\} \quad (7.13)$$

$$\Downarrow$$

$$\left\{ \begin{array}{ll} 1 \text{ massive charged vector field } W_\mu^+(x) & : 2 \times 3 \\ 2 \text{ massive complex scalar fields } \zeta_1^+(x) \text{ and } \zeta_2^+(x) & : 2 \times 2 \end{array} \right\}$$

for the EW up-sector, and

$$\left\{ \begin{array}{ll} 2 \text{ gauge fields } A_\mu^{(2)3}(x) \text{ and } A_\mu^{(1)}(x) & : 2 \times 2 \\ 3 \text{ massless complex scalar fields } \phi_i^0(x) (i = 1, 2, 3) & : 3 \times 2 \end{array} \right\}$$

$$\Downarrow$$

$$\left\{ \begin{array}{ll} 1 \text{ massive real vector field } Z_\mu(x) & : 1 \times 3 \\ 1 \text{ massless electromagnetic field } A_\mu(x) & : 1 \times 2 \\ 1 \text{ massive complex scalar field } \zeta_1^0(x) & : 1 \times 2 \\ 3 \text{ massive real scalar fields } \eta_i(x) (i = 1, 2, 3) & : 3 \times 1 \end{array} \right\} \quad (7.14)$$

for the EW down-sector. Preservation of the number of the independent modes of the fields is confirmed, respectively, as $2 \times 2 + 3 \times 2 = 10 = 2 \times 3 + 2 \times 2$ for the up-sector and $2 \times 2 + 3 \times 2 = 10 = 1 \times 3 + 1 \times 2 + 1 \times 2 + 3 \times 1$ for the down-sector, before and after the phase transition.

At the high energy scale $\check{\Lambda}$, the unitary gauge factor is separated out of the fermion triplets leaving the skeleton triplets $\underline{\Psi}_h^f(x)$ in (6.4). In the low energy region around the scale Λ , the renormalization group effects are presumed to be properly taken into account for all of the quantities in the Lagrangian densities for the Yukawa interactions, (4.5) and (4.6).

Then, through the breakdown of EW symmetry at the scale Λ , the fermion fields acquire masses of Dirac type. Substitution of the decomposition of $\Phi_0(x)$ in (7.4) into (4.5) and (4.6) leads to the effective Lagrangian density for the fermion fields in the low energy region as

$$\mathcal{L}_Y \rightarrow \mathcal{L}_M^Y = \sum_{f=u,d,\nu,e} \bar{\underline{\Psi}}_{L0}^f \mathcal{M}_f \underline{\Psi}_{R0}^f + \text{h.c.} + \dots \quad (7.15)$$

where $\underline{\Psi}_{h0}^f = \Omega_{\text{EW}}(\theta)^{-1} \underline{\Psi}_h^f$, and \mathcal{M}_f are the mass matrices of Dirac type. The ellipsis stands for the interactions of fermion and scalar fields. For the up-sectors ($f = u, \nu$) of EW symmetry, we deduce the Dirac mass matrices as follows:

$$\mathcal{M}_f = a_f I + \frac{1}{3} b_{f1} \begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ -1 & -1 & 2 \end{pmatrix} + \frac{1}{\sqrt{3}} b_{f2} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} + c_f \check{D} \quad (7.16)$$

where $a_f = Y_{f3} v$, $b_{f1} = Y_{f2} v$, $b_{f2} = -Y_{f1} v$ and $c_f = 3Y_{f4} v$. Likewise, for the down-sector ($f = d, e$) of EW symmetry, we obtain

$$\mathcal{M}_f = a_f I + b_{f1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{\sqrt{3}} b_{f2} \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} + c_f \check{D} \quad (7.17)$$

where $a_f = Y_{f3} v$, $b_{f1} = Y_{f1} v$, $b_{f2} = Y_{f2} v$ and $c_f = 3Y_{f4} v$. In each sector, the number of the free parameters is the same with that of the Yukawa coupling constants.

Apparently, these rather peculiar mass matrices are not self-adjoint. To diagonalize such matrices, it is necessary to resort to the bi-unitary transformation⁷⁾

$$V_L^{f\dagger} \mathcal{M}_f V_R^f = \mathcal{M}_{\text{diagonal}}. \quad (7.18)$$

To derive mass eigenvalues, we have to solve the eigenvalue problem for the self-adjoint matrices $\mathcal{M}_f \mathcal{M}_f^\dagger$ as follows:

$$\mathcal{M}_f \mathcal{M}_f^\dagger |\mathbf{v}^{(f)i}\rangle = m_{(f)i}^2 |\mathbf{v}^{(f)i}\rangle. \quad (7.19)$$

The diagonalizing matrix V_L^f is obtained from the eigenvectors. For the charged fermion families ($f = u, d$ and e), solutions of this eigenvalue problems are sufficient to obtain information on mass spectra and diagonalizing matrices. The FMM for quark sector is constructed in the form $V = V_L^{u\dagger} V_L^d = (\langle \mathbf{v}^{(u)i} | \mathbf{v}^{(d)j} \rangle)$.

§8. Discussion

We have developed the gauge field theory of the V and H symmetries which brings forth an effective theory for unified description of flavor physics in low energy regime. The H symmetry generated by the central extension of the Pauli algebra works to reduce uncertainty and create orders in the Yukawa interactions. In particular, the H-sum of the triplet plays essential roles to provide unique structures to the Yukawa interactions.

To make the effective theory to be “effective” in low energy flavor physics, the number of its unknown parameters should be restricted as less as possible. For its purpose, we have formulated the new scheme of the compound mechanism of symmetry breakdown in which the reference state for partially broken symmetry is chosen by specifying the single parameter for the stationary point of the Higgs potential and the freedoms arising in the decomposition of the scalar triplet around the reference state are used to complete the symmetry breakdown and to forbid unphysical modes to appear.

At the high energy scale $\tilde{\Lambda}$, we determined the reference state for the partially broken H symmetry by calculating the stationary-point of the Higgs potential $V_1(\tilde{\Phi})$ and made use of the mixing effect between the component fields of the decomposition of $\tilde{\Phi}(x)$ to exclude the NG mode. At the low energy scale Λ , we had to replace the pseud-reference state which is fixed by the stationary-point of the part of the Higgs potentials $V_2(\Phi) + V_3(\Phi, \tilde{v})$ with the reference state by rescaling the parameter from v_0 to v and utilized the mixing between the component fields of the decomposition of $\Phi(x)$ to suppress both of the NG and tachyonic modes.

The effective theory brought forth by the compound mechanism of symmetry breakdown provides the mass matrices of Dirac type for each sector of basic fermions and the mass matrix of Majorana type for the right-handed chiral neutrinos in the unified way. All of the mass matrices have hierarchical structures. The compound mechanism which is designed not to increase the number of free parameters works successfully to restrict the Dirac mass matrix \mathcal{M}_f of f -sector to possess only four complex numbers. Here it is worth to recognize that multiplication rules of component matrices in each \mathcal{M}_f can reduce the number of independent parameters in $\mathcal{M}_f \mathcal{M}_f^\dagger$. Direct calculation proves that $\mathcal{M}_f \mathcal{M}_f^\dagger$ can be expressed in terms of four real numbers and two phase variables.

In the eigenvalue problems in (7·19) for the quark sector, the eigenvalues for squared masses of the up and down quarks are specified by six real parameters in the mass matrices \mathcal{M}_u and \mathcal{M}_d . Then, we are able to express the FMM composed of the eigenvectors in terms of ten parameters: six eigenvalues, two remaining real parameters and two phase differences. On the other hand, ten different kinds of reliable experimental data, i.e., six mass values and four parameters in the FMM are available in the compilation by Particle Data Group.⁸⁾ Therefore, although no prediction can be made, we are able to apply the present scheme to the data analyses to verify its consistency and to explore hitherto unknown features of quark flavor. Physical situations are much involved for the lepton sector, since the neutrinos

have also the Majorana masses. Recent observations of neutrino oscillations have confirmed that neutrino family possesses minute but non-vanishing squared mass differences.^{9)–11)} One promising way to account for smallness of the neutrino masses is the seesaw mechanism. To determine the neutrino mass spectrum and the FMM for the lepton sector, we must solve the eigenvalue problem for a 6×6 matrix consisting of the Dirac mass matrix \mathcal{M}_ν in (7.16) and the Majorana mass matrix $\tilde{\mathcal{M}}_\nu$ in (6.15). Analyses of the mass matrices for quark and lepton sectors will be made in future investigation.

Our theory predicts the existence of rich physical modes of bose fields. Two and six kinds of bose fields descend, respectively, from the triplets $\check{\Phi}(x)$ and $\Phi(x)$. At the present stage, it is impossible to fix their masses theoretically. It is important, however, to recognize that some of their basic characteristics do not depend on details of the scheme of symmetry breaking. It is, in principle, the representations of the EW and H symmetries that determine patterns of the interactions of the scalar fields with themselves and with other fields. The number of those fields is determined, as shown in the balance sheets in (6.13), (7.13) and (7.14), by the residual degrees of the scalar triplets which are not transferred to the gauge fields.

The real scalar field $\check{\xi}_1(x)$ and $\check{\xi}_2(x)$ descendant from the scalar H triplet $\check{\Phi}(x)$ do not interact with fundamental fermions except for the neutrinos. From (5.2) and (6.7), the squared mass of the would-be NG boson field is derived to be

$$m_{\check{\xi}_1}^2 = \frac{(\check{\lambda}_1 + \check{\lambda}_3)\check{m}_2^2 - (\check{\lambda}_2 + \check{\lambda}_3)\check{m}_1^2}{\check{\lambda}_1 + \check{\lambda}_2 + 2\check{\lambda}_3} \propto \sin \check{\theta}. \quad (8.1)$$

Note that, although it might appear paradoxical, there exist a possibility for the mass $m_{\check{\xi}_1}$ of this boson created around the high energy scale $\check{\Lambda}$ to be comparable to or less than those of the bosonic fields produced around the low energy scale Λ . If this neutral field has the smallest mass among massive boson fields, it may survive a long life through evolution of the universe and give essential influence necessarily on the history of the universe. It is suggestive to interpret the fields $\check{\xi}_1(x)$ and $\check{\xi}_2(x)$ which are blind to the SM quantum numbers might be related with the dark matter and also the dark energy being responsible for the late-time accelerating expansion of the universe.^{12)–15)}

Nine remaining degrees of component fields of the scalar triplet $\Phi(x)$ are survived as six kinds of massive scalar fields: three complex fields $\zeta_1^+(x)$, $\zeta_2^+(x)$ and $\zeta_1^0(x)$; three real fields $\eta_j(x)$ ($j = 1, 2, 3$). The real field $\eta_3(x)$ is just the one corresponding to the Higgs boson in the Weinberg-Salam (WS) theory and all of the remainders are new fields predicted in our theory. It is instructive to consider the case where no rescaling of the reference state is made and no mixing effect exist, i.e., $v = v_0$ ($Z = 1$) and $\theta = 0$. In such a limiting case, two fields $\eta_1(x)$ and $\eta_2(x)$ become massless. Therefore, both of the rescaling and mixing effects are necessary to make those fields to be massive.

Note here that there exist several natural constraints on unknown coupling constants. For the Higgs mechanism can be realized at the low energy scale, the squares of effective Higgs masses, $m_1'^2$ and $m_2'^2$ in (7.6), and the parameter v_0^2 in (5.4) must be positive. These conditions require that the combinations of the coupling con-

stants of mutual-interaction between the triplets $\Phi(x)$ and $\check{\Phi}(x)$, $\dot{\lambda}_1 + \dot{\lambda}_2$, $\dot{\lambda}_3 + \dot{\lambda}_4$ and $(4\dot{\lambda}_1 + \dot{\lambda}_2)/9$ must take smaller values than $(m_1^2 + m_2^2)/v^2$. Contrastingly, it is favorable for those mutual coupling constants $\dot{\lambda}_j$ ($j = 1, \dots, 7$) to have some lower bounds to guarantee that the squared masses in (7·9) and (7·10) are positive and take large values. However, such adjustment of $\dot{\lambda}_j$ is not sufficient to make those squared masses to be positive definite. The relation in (7·11), being independent of the constant $\dot{\lambda}_j$, shows that the coupling constants $\bar{\lambda}_j$ ($j = 1, \dots, 4$) and $\tilde{\lambda}_3$ must also be properly chosen.

The six bose fields can interact with all kinds of the basic fermion fields. However, the experiment establish that the fermions are highly suppressed to communicate with each other through exchanges of neutral scalar fields. This feature known as suppression of the flavor changing neutral current (FCNC) imposes very strong constraints on the masses of the fields $\eta_j(x)$ ($j = 1, 2, 3$) and $\zeta_1^0(x)$. The FCNC bound which results from the $B - \bar{B}$ and $K - \bar{K}$ mixings requires their masses to be larger than $10^2 \sim 10^3 \text{TeV}$.¹⁶⁾ To see implications of this constraint, let us examine the relation in (7·11). Assuming roughly that the constants $\bar{\lambda}_j$ and $\tilde{\lambda}_3$ are almost of the same order and that $Z \sim 10$, we find $\bar{\lambda}v^2 \geq 10^2 \sim 10^4 \text{TeV}^2$. As is well known, from the mass of the weak charged boson $W_\mu^+(x)$ and the Fermi coupling constant G_F , the EW scale parameter v can be estimated by $v^2 \simeq 1/(2\sqrt{2}G_F) \approx 4 \times 10^4 \text{GeV}^2$. Therefore, to satisfy the FCNC constraint, the coupling constants $\bar{\lambda}_j$ and $\tilde{\lambda}_3$ must be larger than $10^4 \sim 10^6$. Compared with the FCNC, the constraints of the flavor changing charged currents are not so stringent. From (7·9) and (7·10), it is unnatural to regard that the charged bosons $\zeta_1^+(x)$ and $\zeta_2^+(x)$ possess the masses of same order with the neutral bosons. It is attractive to consider that a rich spectrum of the scalar fields $\check{\xi}_j(x)$, $\zeta_j^+(x)$, $\zeta_1^0(x)$ and $\eta_j(x)$ could be possible targets of high energy experiments by the LHC group in future.

At this stage, our theory should be evaluated as a provisional scheme of gauge field theory which is designed to result in one effective theory for flavor physics. The symmetries are broken in a special way, called the compound mechanism, which utilizes the freedoms arising in the choice of the reference state and in the decomposition of the scalar triplet around it. Due efforts must be made to justify the compound mechanism of symmetry breakdown in the context of the quantum field theory. We must find physical meanings of the rescaling of the reference state and of the constant $Z(> 1)$.

We have chosen the simple form, in (5·1) and (5·3), for the reference state of partially broken symmetry specified by a single parameter, not to increase the numbers of unknown parameters in the low-energy effective theory. As an extension of our theory, we must examine the reference state which is specified by many parameters and breaks the symmetry totally without any additional assumption. The theory of the private Higgs proposed by Porto and Zee⁵⁾ is an example of many parameters. There may exist such an economical scheme with multi-parameters that can produce a low energy effective theory with handy mass matrices.

The conventional grand unified scheme is framed, independently of the H symmetry, by unifying the vertical symmetry into a simple group. In our scheme, however,

the H symmetry is incorporated into the gauge field theory in a close connection with the EW symmetry. Therefore, we must find a new path to combine the color symmetry with the EW×H symmetry to formulate an extended grand unified theory for particle physics. To avoid the difficulty of quadratic divergence and solve the hierarchy problem, it is requisite to challenge the super-symmetric generalization of the present theory and its extended grand unified version.

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Appendix A

—— *EW×H invariants of scalar fields* ——

In the WS theory, the Higgs potential of the EW doublet $\phi(x)$ possesses the compact form composed solely of the invariant $\phi^\dagger\phi$. This is because the internal space of the EW isospin $\{\tau_a\}$ has simple structure. For ϕ , the dual field can be defined by $\tilde{\phi} = i\tau_2\phi^*$ so that $\tilde{\tilde{\phi}} = \phi$. These fields which have the same transformation property under the EW symmetry group satisfy the relation $\phi^\dagger\tilde{\phi} = 0$. The “internal vector” $\phi^\dagger\tau_a\phi$ satisfy a simple identical equation, i.e., its squared length can be expressed in terms of the pure “internal scalar” quantity as follows:

$$(\phi^\dagger\tau_a\phi)(\phi^\dagger\tau^a\phi) = (\phi^\dagger\phi)^2. \quad (\text{A}\cdot 1)$$

In contrast to the WS theory, the internal space of EW×H symmetry has complex structure. First of all, it is impossible to define the *dual* fields for the triplets $\tilde{\Phi}(x)$ and $\tilde{\tilde{\Phi}}(x)$. For example, the associated field for $\tilde{\tilde{\Phi}}(x)$ defined in (3.5) can not be *dual* to $\tilde{\Phi}(x)$, since

$$\tilde{\tilde{\tilde{\Phi}}}(x) = -(I - \tilde{D})\tilde{\Phi} \neq \tilde{\Phi}. \quad (\text{A}\cdot 2)$$

Nevertheless, it is possible to prove that, as in (A.1) in the SM, all of the quartic invariants composed of internal vectors and bi-vectors are expressed in terms of the pure “internal scalar” quantities. The proof is given below in three steps.

Some of the identical equations can be proved directly by the definitions. For example, the H-sums of the associated scalar triplets $\tilde{\tilde{\Phi}}(x)$ and $\tilde{\Phi}(x)$ vanish, i.e.,

$$\{\{\tilde{\tilde{\Phi}}\}\} = 0, \quad \{\{\tilde{\Phi}\}\} = 0 \quad (\text{A}\cdot 3)$$

due to the action of $\tilde{\tau}_2$. To examine and classify the invariants in general, however, it is necessary to use explicit representations of the scalar triplets in terms of the eigenvectors in (2.9):

$$\tilde{\Phi} = z_1|1\rangle + z_2|2\rangle + z_3|3\rangle, \quad \tilde{\tilde{\Phi}} = z_2^*|1\rangle - z_1^*|2\rangle \quad (\text{A}\cdot 4)$$

where $z_j(x)$ are complex scalar fields, and

$$\Phi = Z_1|1\rangle + Z_2|2\rangle + Z_3|3\rangle, \quad \tilde{\Phi} = i\tau_2 Z_2^*|1\rangle - i\tau_2 Z_1^*|2\rangle \quad (\text{A}\cdot 5)$$

where $Z_j(x) = {}^t(\zeta_j^+, \zeta_j^0)$ are complex scalar fields of EW doublets.

Using the decompositions in (A.4), the triplets $\check{\Phi}(x)$ and $\check{\tilde{\Phi}}(x)$ are proved to satisfy the identity

$$\check{\Phi}^\dagger \check{\tilde{\Phi}} = 0. \quad (\text{A.6})$$

It is this identity that simplifies the quartic relations of the triplet $\check{\tilde{\Phi}}$. The bilinear form of the associated triplet $\check{\tilde{\Phi}}$ is reduced to those of the triplet $\check{\Phi}$ as

$$\check{\tilde{\Phi}}^\dagger \check{\tilde{\Phi}} = \check{\Phi}^\dagger (I - \check{D}) \check{\Phi} = \check{\Phi}^\dagger \check{\Phi} - \frac{1}{3} \{\check{\Phi}^\dagger\} \{\check{\Phi}\}. \quad (\text{A.7})$$

Quartic relations of the triplets satisfy the following identical equations as

$$\left(\check{\Phi}^\dagger \check{\tau}_j \check{\tilde{\Phi}} \right) \left(\check{\tilde{\Phi}}^\dagger \check{\tau}^j \check{\Phi} \right) = 2 \left(\check{\Phi}^\dagger (I - \check{D}) \check{\Phi} \right)^2 \quad (\text{A.8})$$

and

$$\left(\check{\Phi}^\dagger \check{\tau}_j \check{\Phi} \right) \left(\check{\Phi}^\dagger \check{\tau}^j \check{\Phi} \right) = \left(\check{\Phi}^\dagger (I - \check{D}) \check{\Phi} \right)^2 = \left(\check{\Phi}^\dagger \check{\Phi} - \frac{1}{3} \{\check{\Phi}^\dagger\} \{\check{\Phi}\} \right)^2. \quad (\text{A.9})$$

These quartic relations show that invariant combinations including internal vectors composed of scalar triplet-spinors are reducible to invariant quantities consisting of $\check{\Phi}^\dagger \check{\Phi}$ and $\{\check{\Phi}^\dagger\} \{\check{\Phi}\}$.

In contrast with the identity in (A.6), $\Phi(x)$ and $\tilde{\Phi}(x)$ are *not orthogonal* with each other. The decompositions in (A.5) leads readily to

$$\Phi^\dagger \tilde{\Phi} = Z_1^\dagger i\tau_2 Z_2^* - Z_2^\dagger i\tau_2 Z_1^* = 2(\zeta_1^- \zeta_2^{0*} - \zeta_2^- \zeta_1^{0*}) \neq 0. \quad (\text{A.10})$$

This turns out to be a main cause of complexity of the Higgs potential $V_2(\Phi)$ in (4.15). The triplets $\Phi(x)$ and $\tilde{\Phi}(x)$ satisfy the following quadratic relations as

$$\tilde{\Phi}^\dagger \tilde{\Phi} = {}^t\Phi(I - \check{D})\Phi^* = \Phi^\dagger \Phi - \frac{1}{3} \{\Phi^\dagger\} \{\Phi\}, \quad (\text{A.11})$$

$$\Phi^\dagger \check{\tau}_j \tilde{\Phi} = 0, \quad \Phi^\dagger \tau_a \tilde{\Phi} = 0, \quad (\text{A.12})$$

$$\tilde{\Phi}^\dagger \check{\tau}_j \tilde{\Phi} = -\Phi^\dagger \check{\tau}_j \Phi, \quad \tilde{\Phi}^\dagger \tau_a \tilde{\Phi} = -\Phi^\dagger \tau_a \Phi. \quad (\text{A.13})$$

By using the decomposition in (A.5), we are able to prove the following quartic relations of the scalar triplets as

$$\left(\Phi^\dagger \check{\tau}_j \Phi \right) \left(\Phi^\dagger \check{\tau}^j \Phi \right) = |\Phi^\dagger (I - \check{D}) \Phi|^2 - |\Phi^\dagger \tilde{\Phi}|^2, \quad (\text{A.14})$$

$$\left(\Phi^\dagger \tau_a \Phi \right) \left(\Phi^\dagger \tau^a \Phi \right) = |\Phi^\dagger \Phi|^2 + 2\Phi^\dagger i\tau_2 {}^t\Phi^* \Phi i\tau_2 \Phi, \quad (\text{A.15})$$

$$\left(\Phi^\dagger \tau_a (I - \check{D}) \Phi \right) \left(\Phi^\dagger \tau^a (I - \check{D}) \Phi \right) = |\Phi^\dagger (I - \check{D}) \Phi|^2 - |\Phi^\dagger \tilde{\Phi}|^2, \quad (\text{A.16})$$

$$\left(\Phi^\dagger \tau_a \check{D} \Phi \right) \left(\Phi^\dagger \tau^a \check{D} \Phi \right) = |\Phi^\dagger \check{D} \Phi|^2 = \frac{1}{9} \left(\{\Phi^\dagger\} \{\Phi\} \right)^2, \quad (\text{A.17})$$

$$\left(\Phi^\dagger \tau_a \check{\tau}_j \Phi\right) \left(\Phi^\dagger \tau^a \check{\tau}^j \Phi\right) = |\Phi^\dagger (I - \check{D}) \Phi|^2 + 2|\Phi^\dagger \check{\Phi}|^2, \quad (\text{A}\cdot 18)$$

$$\left(\Phi^\dagger \tau_a \check{\tau}_j \check{\Phi}\right) \left(\check{\Phi}^\dagger \tau^a \check{\tau}^j \Phi\right) = 4|\Phi^\dagger (I - \check{D}) \Phi|^2 - |\Phi^\dagger \check{\Phi}|^2. \quad (\text{A}\cdot 19)$$

With these identical relations, all invariants composed of internal vectors and bi-vectors composed of the triplets of EW doublets can be reduced to the “internal scalar” invariants composed.

There are two identical relations concerning the internal vectors of the triplets $\check{\Phi}(x)$ and $\Phi(x)$ as

$$(\check{\Phi}^\dagger \check{\tau}_j \Phi)^\dagger (\check{\Phi}^\dagger \check{\tau}^j \Phi) = (\check{\Phi}^\dagger (I - \check{D}) \check{\Phi}) (\Phi^\dagger (I - \check{D}) \Phi) + (\Phi^\dagger \check{\Phi})^\dagger (\Phi^\dagger \check{\Phi}) \quad (\text{A}\cdot 20)$$

and

$$(\check{\Phi}^\dagger \check{\tau}_j \check{\Phi})^\dagger (\Phi^\dagger \check{\tau}^j \Phi) = (\check{\Phi}^\dagger (I - \check{D}) \check{\Phi}) (\Phi^\dagger (I - \check{D}) \Phi) - 2(\Phi^\dagger \check{\Phi})^\dagger (\Phi^\dagger \check{\Phi}). \quad (\text{A}\cdot 21)$$

Evidently, the triplets $\check{\Phi}(x)$ and $\Phi(x)$ possess quartic invariants $|\Phi^\dagger (I - \check{D}) \check{\Phi}|^2$ and $|\Phi^\dagger \check{D} \check{\Phi}|^2$. Further, there exist intricate terms such as $(\Phi^\dagger \check{\Phi}) (\check{\Phi}^\dagger \Phi)$, $(\Phi^\dagger \check{\Phi}) (\check{\Phi}^\dagger \check{\Phi})$, $(\check{\Phi}^\dagger \Phi) (\check{\Phi}^\dagger \check{\Phi})$ and $(\check{\Phi}^\dagger \check{\Phi}) (\check{\Phi}^\dagger \check{\Phi}) = (\Phi^\dagger \check{\Phi}) (\check{\Phi}^\dagger \Phi)$. However, the conservation of the H hypercharge and the assignment of $\check{y}_{\check{\Phi}} \neq 0$ and $\check{y}_{\Phi} = 0$ prohibit these terms, except for the last one, to appear in the Higgs potential $V_3(\Phi, \check{\Phi})$.

With resort to the identical relations proved in this Appendix, it turns out possible to express all EW×H invariants in terms of the internal “scalar” quantities.

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